PG Part–I (DE) Exam., 2016

2016

MATHEMATICS

Paper : I

(Unit—I)

Time : 2 hours | *Full Marks* : 45

The figures in the margin indicate full marks.

SECTION—A

Answer any *two* of the following : 9×2=18

1. Let $f : G \to G^*$ be a group epimorphism. Let H be a normal subgroup of G such that $H \subseteq \ker f$ and g be the natural homomorphism of G onto $G|H$. Then prove that $\exists$ a unique homomorphism h from $G|H$ onto G^* such that $f = h \circ g$. Further, prove that h is one-one iff $H = \ker f$. 7+2

2. *(a)* State and prove Zassenhaus lemma.

   (b) Define a solvable series in a group. 6+3

3. *(a)* Define class equation of a finite group.

(b) Employ the class equation to prove that if G be a group of order n such that n is divisible by a prime p, then G contains an element of order p. 2+7

SECTION—B

Answer any *three* questions : 6×3=18

4. Prove that every infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$. 6

5. Prove that every group of order 96 has a normal subgroup of order 16 or 32. 6

6. State and prove Cayley-Hamilton theorem. 6

7. Given a ring R with 1 and a non-empty set S, construct a free R-module on S. 6

8. Let H and K be two subgroups of a group G. Prove that the number of distinct conjugates of H induced by the elements of K is equal to $[K : N_K(H)]$, the index of $N_K(H)$ in K. 6

SECTION—C

Answer *all* the questions : 3×3=9

9. Define a homomorphism $f : GL(2, \mathbb{R}) \to \mathbb{R}^*$ which is onto but not one-one, where

$$GL(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}; a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$$

and $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. 3

10. Give an example of two equivalent subnormal series of a group. 3

11. Prove that every group of order p^2 is commutative, p being a prime. 3

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PG Part–I (DE) Exam., 2016

2016

MATHEMATICS

Paper : I

(Unit–2)

Time : 2 hours | *Full Marks* : 45

The figures in the margin indicate full marks.

Unless otherwise stated, notations carry their usual meanings.

SECTION—A

Answer any **two** of the following : 9×2=18

1. Show that the ring of Gaussian integers $\mathbb{Z}[i]$, is a Euclidean domain.

2. Let $F|K$ be a finite extension. Prove that $|\text{Aut}(F|K)| = [F:K]$ iff $F|K$ is normal and separable.

3. Prove that if D is a unique factorization domain, so is $D[x]$.

SECTION—B

Answer *any* **three** questions : 6×3=18

4. Show that the ring $\mathbb{Z}(\sqrt{3})$ and $\mathbb{Z}(\sqrt{5})$ are not isomorphic, where $\mathbb{Z}(\sqrt{3}) = \{a + b\sqrt{3};\ a, b \in \mathbb{Z}\}$ and $\mathbb{Z}(\sqrt{5}) = \{a + b\sqrt{5};\ a, b \in \mathbb{Z}\}$.

5. If $K(u)$ be a simple algebraic extension of a field K, then show that $K(u) = K[u]$, where $K[u] = \{a_0 + a_1 u + \cdots + a_n u^n;\ a_i \in k,\ i = 0, 1, 2, \cdots n\}$.

6. Let p be a prime and n be a positive integer. If F be a field with $|F| = p^n$, then prove that F is the splitting field of $f(x) = x^{p^n} - x \in \mathbb{Z}_p[x]$.

7. Prove that in a principal ideal domain, an ideal is a prime ideal iff it is a maximal ideal.

8. Let $f(x)$ be an irreducible polynomial over a field K. Prove that $f(x)$ is separable over K iff $f(x)$ and $f'(x)$ are relatively prime.

SECTION—C

Answer **all** questions : 3×3=9

9. Show that [3] is not irreducible in $\mathbb{Z}_{12}$.

10. Show that $\mathbb{C}$ is an algebraic extension of $\mathbb{R}$.

11. Show that every field of characteristic 0 is perfect.

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PG Part–I (DE) Exam., 2016

2016

MATHEMATICS

Paper : II

(Unit–1)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

SECTION—A

Answer *any* **one** question.

1. *(a)* Let E have finite measure, $f_n \to f$ in measure on E and g be a measurable function on E that is finite a.e. on E. Prove that $\{f_n \cdot g\} \to f \cdot g$ in measure. Use it to show that $\{f_n^2\} \to f^2$ in measure. 9

(b) Let E be a measurable set of finite outer measure. Prove that for each $\varepsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^{n}$ for which if $O = \bigcup_{k=1}^{n} I_k$, then $m^*(E \setminus O) + m^*(O \setminus E) < \varepsilon$. 9

2. *(a)* Let f and ϕ be bounded functions defined on $[a, b]$. Suppose that f is continuous on $[a, b]$, ϕ is differentiable on $[a, b]$ and ϕ' is Riemann integrable on $[a, b]$. Prove that f is Riemann-Stieltjes integrable on $[a, b]$ w.r.t. ϕ and

$$(RS)\int_a^b f d\phi = (R)\int_a^b f\phi'$$

9

(b) Given a measure space (X, S, μ). Let f be a non-negative extended real-valued S-measurable function on X. If the set function υ on S is defined by

$$\upsilon(E) = \int_E f d\mu;\ E \in S$$

then prove that υ is a measure on (X, S). 9

SECTION—B

Answer *any* **three** questions.

3. Suppose f, g are convex on an interval I. Prove that $f + g$ and cf are convex on I for any $c \geq 0$.

6

4. Let $f, g \in BV[a, b]$. Show that $fg \in BV[a, b]$ and

$$V_a^b(fg) \leq \sup_{[a, b]} |f| \cdot V_a^b(g) + \sup_{[a, b]} |g| \cdot V_a^b(f)$$ 6

5. Let $A \subset \mathbb{R}$ and B be a measurable subset of $\mathbb{R}$. Prove that—

(a) there exists a G_δ-set G such that $A \subset G$ and $m(G) = m^*(A)$;

(b) if $m(B) < \infty$ and $A \subset B$, then A is measurable if and only if

$$m(B) = m^*(A) + m^*(B \setminus A)$$ 3+3=6

6. Show that for every $E \subset \mathbb{R}$ and $\varepsilon > 0$ there exists an open set O in $\mathbb{R}$ such that $O \supset E$ and

$$m^*(E) \leq m^*(O) \leq m^*(E) + \varepsilon$$ 6

7. Let $f : E \to \mathbb{R}$ be a measurable function. Prove that $\{x \in E : f(x) = r\}$ is a measurable set for each real number r. Give an example to show that the converse is false. 6

8. Let f and ϕ be bounded functions defined on $[a, b]$. Suppose that f and ϕ have a common point of discontinuity in $[a, b]$. Prove that f is not RS-integrable w.r.t. ϕ on $[a, b]$. 6

SECTION—C

Answer **all** questions.

9. Let f be a real-valued function on $[0, 2\pi]$ defined by

$$f(x) = \sin\frac{1}{x} \quad \text{for} \quad x \in (0, 2\pi]$$
$$= 0 \quad \text{for} \quad x = 0$$

Show that $f \notin BV[0, 2\pi]$. 3

10. Prove that every closed interval $[a, b] \subset \mathbb{R}$ is a G_δ. 3

11. Suppose that E is a Lebesgue measurable set in $\mathbb{R}$ with finite measure. Prove that there is a measurable set $A \subset E$ such that $m(A) = m(E)/2$. 3

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PG Part–I (DE) Exam., 2016

2016

MATHEMATICS

Paper : II

(Unit–2)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

Notations and symbols have their usual meanings, unless stated otherwise.

Throughout E denotes an open subset of R^n.

Answer **two** from Section—A, **three** from Section—B and **all** from Section—C.

SECTION—A

1. Suppose f is a C^1-mapping on E into R^n with $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$. Prove that there exist open sets U and V in R^n such that $a \in U$, $b \in V$, f is one-to-one on U and $V = f(U)$. 9

2. If $f : R^m \to R$ be a function of class C^2, then prove that, for each $a \in R^m$,

$$D_i D_j f(a) = D_j D_i f(a)$$ 9

3. Suppose T is a C^1-mapping on E into an open set $V \subset R^m$, S is a C^1-mapping of V into an open set $W \subset R^p$, and w is a k-form in W, so that w_S is a k-form in V and both $(w_S)_T$ and w_{ST} are k-forms in E where ST is defined by $(ST)(x) = S(T(x))$. Prove that $(w_S)_T = w_{ST}$. 9

SECTION—B

4. Define directional derivative of $f : E \to R^m$. Show that the function $f : R^2 \to R$ defined by

$$f(x, y) = \frac{x^2 y}{x^4 + y^2} \quad \text{if} \quad (x, y) \neq (0, 0)$$
$$= 0 \quad \text{if} \quad (x, y) = (0, 0)$$

is not continuous at (0, 0) but the directional derivative at (0, 0) exists in all directions. 6

5. If X is a compact subspace of R^n, then prove that X is closed and bounded. 6

6. Let $f : E \to R^n$ be of class C^1. Assume that $Df(x)$ is non-singular. Show that if f is one-to-one on E, the set $B = f(E)$ is open in R^n. 6

7. Let $f : R^5 \to R^2$ be given by

$$f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2 y_1 - 4y_2 + 3$$

$$f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos x_1 - 6x_1 + 2y_1 - y_3$$

and $f = (f_1, f_2)$. Find $f'(a, b)$ for $a = (0, 1)$ and $b = (3, 2, 7)$. Show that there exists a C^1-mapping g defined in a *nbd* of (3, 2, 7) such that $g\ (3, 2, 7) = (0, 1)$ and $f(g(y), y) = 0$. Hence find $g'(3, 2, 7)$. 6

8. Let $w = \sum_I b_I(x)\, dx_I$ be the standard presentation of a k-form w in E. If $w = 0$ in E, then prove that $b_I(x) = 0$ for every increasing k-index I and for every $x \in E$. 6

SECTION—C

9. Prove that for $f : E \to R^m$ and $x \in E$, $f'(x)$ is unique. 3

10. Find the critical points, relative extremum and saddle points of $f : R^2 \to R$ given by $f(x, y) = x^3 + x - 4xy + 2y^2$. 3

11. For a k-form w of class C^2 in E, prove that $d^2 w = 0$. 3

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PG Part–I (DE) Exam., 2016

2016

MATHEMATICS

Paper : III

(Unit–1)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

Answer **two** from Section—A, **three** from Section—B and **all** from Section—C.

SECTION—A

1. Let α and β be any two cardinal numbers of which α is infinite and $\beta \neq 0$. Then show that $\alpha\beta = \alpha$ if $\alpha \geq \beta$. 9

2. Consider an operator $K : P(X) \to P(X)$, which satisfies the following conditions :

(i) $K(A) \supset A, \quad \forall A \in P(X)$

(ii) $K(\phi) = \phi$

(iii) $K(A \cup B) = K(A) \cup K(B), \quad \forall A, B \in P(X)$

(iv) $K(K(A)) = K(A), \quad \forall A \in P(X)$

Show that $J(K) = \{A \subset X : K(X - A) = X - A\}$ is a topology on X and $\overline{A} = K(A), \; \forall A \in P(X)$, where $\overline{A}$ is the closure of A in $(X, J(K))$. 9

3. Show that a T_1 topological space X is T_4, if and only if every continuous function $f : C \to \mathbb{R}$, C is closed subspace of X, can be extended to X continuously. 9

4. State and prove Schroder-Bernstein theorem. 9

SECTION—B

5. Show that a topological space X is regular if and only if for any $x \in X$ and any open set $U \ni x$, $\exists$ an open set V such that $x \in V$ and $\overline{V} \subset U$. 6

6. Prove that a mapping $f : X \to Y$ is continuous iff for any $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$. 6

7. Let β be a family of subsets of X satisfying *(i)* $\forall\, x \in X \;\exists B \in \beta$, so that $x \in B$ and *(ii)* $B_1, B_2 \in \beta$, $x \in B_1 \cap B_2 \Rightarrow \exists\, B_3 \in \beta$ with $x \in B_3 \subset B_1 \cap B_2$. Show that β forms a basis for some topology on X. 6

8. Show that in a topological space X, for any $A \subset X$

(i) $\mathring{\mathring{A}} = \mathring{A}$

(ii) $\overline{\overline{A}} = \overline{A}$

(iii) $(\bar{A})^{c} = (A^{c})^{\circ}$

(iv) $\overline{A^{c}} = (A^{\circ})^{c}$

where c stands for complementation. 6

9. Show that $f:(0,1)\to\mathbb{R}$ defined by $f(x)=\dfrac{1-2x}{x(1-x)}$ is a homeomorphism. 6

10. Show that $\mathbb{R}$ with lower limit topology is not second countable but separable. 6

SECTION—C

11. Show that $\mathbb{R}_l$ is T_4. 3

12. Find 2^a+5a+2^c, where $|\mathbb{N}|=a$ and $|\mathbb{R}|=c$. 3

13. Is there any finite T_1 topological space which is not discrete? Give reason. 3

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PG Part–I (DE) Exam., 2016

2 0 1 6

MATHEMATICS

Paper : III

(Unit–2)

Time : 2 hours | *Full Marks* : 45

The figures in the margin indicate full marks.

Answer **two** from Section—A, **three** from Section—B and **all** from Section—C.

SECTION—A

1. Show that union of a family of connected subsets of a topological space X is connected if the connected subsets have a common point. Also show that the above statement is sufficient but not necessary. 9

2. Show that the product of a paracompact space and a compact space is paracompact. 9

3. Define one-point compactification of a locally compact (non-compact) T_2 space. Show that one-point compactification of $\mathbb{R}^2$ is $\mathbf{C}$. 9

4. Show that compact subset of a T_2 space is closed. Give an example to show that a compact subset may not be closed. 9

SECTION—B

5. Show that every path-connected space is connected. Is the converse true? Give example. 6

6. Show that a net is convergent iff the associated filter is convergent and converges to same limit. 6

7. Show that paracompact T_2 space is T_4. 6

8. If $f : X \to Y$, X is compact, Y is T_2 and f is a bijection, then show that f is a homeomorphism. 6

9. Show that a function $f : X \to Y$, X and Y are topological spaces, is continuous at a point a iff for every net $\{x_\alpha\}_{\alpha \in \Lambda}$ in X converges to a, the image net $\{f(x_\alpha)\}_{\alpha \in \Lambda}$ converges to $f(a)$. 6

10. Show that the product of regular spaces is regular. 6

SECTION—C

11. Find the connected and path-connected components of $\mathbb{R}_l$. 3

12. Find two-point compactification of $\mathbb{R}$. 3

13. Examine if $\mathbb{R}$ with cofinite topology is compact. 3

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PG Part–I (DE) Exam., 2016

2016

MATHEMATICS

Paper—IV

(Unit—1)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

Unless otherwise stated, notations carry their usual meanings.

GROUP—A

Answer *all* questions : 3×3=9

1. If f is analytic in a region and if $|f|$ is constant there, then prove that f is constant.

2. Suppose $\{f_n\}$ represents a sequence of functions, analytic in an open domain D and such that $f_n \to f$ uniformly on every compact subset $K \subset D$. Show that f is analytic in D.

3. Evaluate $\int_\gamma z^2\, dz$, where $\gamma : z = t - it^2, 0 \le t \le 1$.

GROUP—B

Answer any *three* questions : 6×3=18

4. State and prove Liouville's theorem. 6

5. Prove that, if t is entire and $f(z) \to \infty$ as $z \to \infty$, then f is a polynomial. 6

6. Evaluate $\int_{|z|=2} \frac{dz}{(z-4)(z^3-1)}$, using residue theorem. 6

7. Define cross-ratio. Show that the cross-ratio of four points is invariant under bilinear transformation. 6

8. Suppose that f is analytic in a region D and that $f(z_n)=0$, where $\{z_n\}$ is a sequence of distinct points and $z_n \to z_0 \in D$. Then prove that $f \equiv 0$ in D. 6

GROUP—C

Answer any *two* questions : 9×2=18

9. Suppose f is entire and Γ is the boundary of a rectangle R. Then show that $\int_\Gamma f(z)\,dz = 0$.

10. *(a)* Find the number of zeros of $f(z) = z^4 - 5z + 1$ in $1 \leq |z| \leq 2$.

(b) Evaluate $\int_0^\infty \frac{dx}{1+x^3}$. 5+4=9

11. *(a)* Find a conformal mapping f of the semidisc $s = \{z : |z| < 1, \text{Im}(z) > 0\}$ onto the upper half-plane.

(b) Suppose that f is analytic and bounded by 1 in the unit disc and that $f(\frac{1}{2}) = 0$. Find the maximum value of $|f(\frac{3}{4})|$. 5+4=9

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PG Part–I (DE) Exam., 2016

2 0 1 6

MATHEMATICS

Paper : IV

(Unit–2)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

Unless otherwise stated, notations carry their usual meanings.

GROUP—A

Answer **all** questions : 3×3=9

1. If u is harmonic in a simply connected domain D, then prove that u is the real part of an analytic function in D.

2. Prove that

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right)$$

is convergent.

3. Let f be analytic in the cut plane $D = \mathbb{C} \setminus (-\infty, 0)$ such that $f(x) = x^x$ for all $x > 0$. Show that $f(z) = \overline{f(\bar{z})}$ for all $z \in D$.

GROUP—B

Answer *any* **three** questions : 6×3=18

4. Prove that the series

$$\sum_{k=1}^{\infty} \log z_k$$

and the product $\prod_{k=1}^{\infty} z_k$ either converge or diverge together.

5. Show that every mesomorphic function in $\mathbb{C}$ can be represented as a quotient of two entire functions.

6. Prove that

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|r - ae^{i\theta}|\, d\theta = \ln r,$$

where $a \in \mathbb{C}$ and $|a| < r$.

7. State and prove Harnack's Inequality for the unit disk.

8. Suppose f is analytic at z_0 and $f'(z_0) \neq 0$. Prove that f is a conformal mapping.

GROUP—C

Answer *any* **two** questions : 9×2=18

9. Show that

$$\pi \csc \pi z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2(-1)^k z}{z^2 - k^2}, z \in \mathbb{C} \setminus \mathbb{Z}$$

10. Prove that for all $s \in \mathbb{C}$, the ζ-function satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$$

11. Let f be an entire function of finite order λ. Suppose that $a_1, a_2, \cdots,$ are the zeros of $f(z)$ listed with multiplicities and $0 < |a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots$. Show that f has a finite genus μ satisfying $\mu \le \lambda$.

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PG Part–I (DE) Exam., 2016

2016

MATHEMATICS

Paper : V

(Unit—1)

Time : 2 hours | *Full Marks* : 45

The figures in the margin indicate full marks.

Notations and symbols have their usual meanings, unless stated otherwise.

GROUP—A

Answer any *two* questions : 9×2=18

1. (a) Define Lie bracket. 1

(b) Let M be a smooth manifold. For any $X, Y, Z \in \chi(M)$ and $f, g \in C^{\infty}(M)$, prove the following : 8

(i) $[X, Y] = -[Y, X]$

(ii) $[X, X] = 0$

(iii) $[X, Y + Z] = [X, Y] + [X, Z]$

(iv) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

2. What do you mean by differentiable manifold of dimension n? Show that S^n is an n-dimensional differentiable manifold. 3+6=9

3. *(a)* Define exterior derivative. 2

(b) If w is a 1-form on a smooth manifold M, then prove that the exterior derivative dw satisfies

$$dw(fX, Y) = f\,dw(X, Y)$$

$\forall\, X, Y \in \chi(M)$ and $\forall\, f \in C^\infty(M)$. 7

4. *(a)* Define Lie group isomorphism. 2

(b) Let $f : G \to H$ is a Lie group homomorphism. Then prove that the induced map $f_* : T_e G \to T_{e'} H$ is a homomorphism between the Lie algebra of the groups. Also prove that if f is a Lie group isomorphism, then f_* is an isomorphism between the Lie algebras, where $e' = f(e) \in H$, for $e \in G$. 7

GROUP—B

Answer any *three* questions : 6×3=18

5. If $X, Y \in \chi(M)$, then prove that $[X, Y] \in \chi(M)$, where M is a smooth manifold. 6

6. If w is a 1-form on a smooth manifold M of dimension n, then show that

$$dw(X, Y) = \frac{1}{2}\{X\,w(Y) - Y\,w(X) - w([X, Y])\}$$

for any $X, Y \in \chi(M)$; dw is a 2-form on M. 6

7. Let G be a Lie group and $\phi: G \to G$ be a mapping such that $\phi(a) = a^{-1}$, $a \in G$. Then prove that a 1-form w is left invariant if and only if $\phi^* w$ is right invariant. 6

8. What do you mean by distribution? When is a distribution said to satisfy the Frobenius condition? 4+2=6

9. *(a)* Is the map $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, a chart? Justify your answer. 3

(b) Compute the exterior product $(5\,du^1 + 3\,du^2) \wedge (3\,du^1 + 2\,du^2)$. 3

10. *(a)* When is a differential form w called closed and when is it called exact? 2

(b) Determine which of the following differential forms on $\mathbf{R}^3$ is closed and which is exact : 4

(i) $w = yzdx + xzdy + xydz$

(ii) $\eta = xdx + x^2y^2dy + yzdz$

GROUP—C

Answer *all* questions : 3×3=9

11. What is diffeomorphism? Give an example. 3

12. If X and Y are differentiable vector fields on $\mathbb{R}^3$ defined by $X = \frac{\partial}{\partial x^1}$, $Y = \frac{\partial}{\partial x^2} + e^{x^1} \frac{\partial}{\partial x^3}$, then compute $[X, Y]$. 3

13. Determine the integral curve for the vector field in $\mathbb{R}^2$ given by $X = x^2 \frac{\partial}{\partial x^1} - (x^2)^3 \frac{\partial}{\partial x^2}$. 3

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PG Part–I (DE) Exam., 2016

2016

MATHEMATICS

Paper—V

(Unit—2)

Time : 2 hours *Full Marks* : 45

The figures in the margin indicate full marks.

Notations and symbols have their usual meanings, unless stated otherwise.

GROUP—A

Answer any *two* questions : 9×2=18

1. If R is the Riemannian curvature tensor of a Riemannian manifold (M, g), then prove that—

(a) $g(R(X, Y)Z, U) = -g(R(X, Y)U, Z)$;

(b) $g(R(X, Y)Z, U) = g(R(Z, U)X, Y)$;

$\forall U, X, Y, Z \in \chi(M)$. 9

2. Suppose (M^n, g) is a Riemannian manifold with the Levi-Civita connection $\overline{\nabla}$. If the manifold admits another metric connection ∇ with non-vanishing torsion T, then prove that for any $X, Y, Z \in \chi(M)$, all have

$$\nabla_X Y - \overline{\nabla}_X Y = \frac{1}{2}\{T(X, Y) + T'(X, Y) + T'(Y, X)\}$$

where $g(T(Z, X), Y) = g(T'(X, Y), Z)$. 9

3. Deduce the equations of Gauss, Codazzi and Ricci. 9

4. *(a)* Define an almost complex manifold and a complex manifold. 4

(b) State and prove second Bianchi identity. 5

GROUP—B

Answer any *three* questions : 6×3=18

5. *(a)* Write down Koszul's formula. 1

(b) State the fundamental theorem of Riemannian geometry. 2

(c) Define Ricci tensor, scalar curvature and Gaussian curvature. 3

6. If $T(X, Y)$ is the torsion tensor of the affine connection ∇, then prove that—

(a) $T(X, Y)$ is R-bilinear;

(b) $T(X, Y)$ is $C^{\infty}(M)$-bilinear;

$\forall\ X, Y \in \chi(M)$ where M is a smooth manifold. 6

7. *(a)* What is Levi-Civita connection? 2

(b) State and prove first Bianchi identity. 4

8. Prove that any affine connection ∇ on a smooth manifold M can be decomposed into a sum of a multiple of its torsion tensor and a torsion-free connection. 6

9. State and prove Ricci identity. 6

10. If X and Y are killing vector fields on a Riemannian manifold (M, g), then so is $[X, Y]$. Prove it. 6

GROUP—C

Answer *all* questions : 3×3=9

11. Write a short note on conformal transformation.

12. Write down Gauss formula and Weingarten formula.

13. Define torsion tensor and show that it is skew-symmetric.

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