DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTERS OF SCIENCE-MATHEMATICS

SEMESTER -I

ABSTRACT ALGEBRA

DEMATH-1 CORE-1

BLOCK-1

UNIVERSITY OF NORTH BENGAL

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First Published in 2019

FOREWORD

The Self-Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

ABSTRACT ALGEBRA

BLOCK-1

BLOCK-2

BLOCK-1: ABSTRACT ALGEBRA

In this block we will go through

Unit I :In this unit we will discuss about Homomorphisms, Isomorphisms, Group Isomorphism various properties of those functions between groups which preserve the algebraic structure of their domain groups.

Unit II: In this unit we will discuss about Fundamental Theorem of Homomorphism, Automorphisms After understanding the concept of isomorphisms & result about the relationship between homomorphism's and quotient groups is the Fundamental Theorem of Homomorphism for groups

Unit III: In this unit we will discuss about Groups, Symmetric Group, Cyclic Decomposition, Alternating Group, Cayley's Theorem, symmetric groups and their subgroups are called permutation groups The study of permutation groups and groups of transformations that gave the foundation to group theory a result by the mathematician Cayley, which says that every group is isomorphic to permutations group, This result is what makes permutation groups

Unit IV: In this unit we will discuss about Direct Product of Groups, External Direct Product, Internal Direct Product, Introduction To Sylow Theorems, Groups of Order, Finite Abelian Groups All cyclic groups are finite abelian but a finite abelian group is not necessarily cyclic & All subgroups of a finite abelian group are normal.

Unit V: In this unit we will discuss about algebra be attached in group actions In this unit getting the information related to conjugate elements

Unit VI: In this unit we will discuss about Cauchy-Riemann equations which under certain conditions provide the necessary and sufficient condition for the differentiability of a function of a complex variable at a point A very important concept of analytic functions which is useful in many application of the complex variable theory & discuss the concept of Cauchy's theorem

Unit VI: In this unit we will discuss about the Sylow Theorems provide a partial converse for Lagrange's Theorem: in certain cases they guarantee us subgroups of specific orders. These theorems yield a powerful set of tools for the classification of all finite non-abelian groups.

UNIT – 1: HOMOMORPHISM OF GROUPS

STRUCTURE

1.0 OBJECTIVES

After studying this unit, you should be able to:

- Explain the concept of homomorphism
- Describe Isomorphism

1.1 INTRODUCTION

In this unit, we will discuss various properties of those functions between groups which preserve the algebraic structure of their domain groups. These functions are called group Homomorphisms. This term was introduced by the mathematician Klein in 1983

In this unit, you will also get an idea about a very important mathematical ideaisomorphism

1.2 HOMOMORPHISMS

Let us start our study of functions from one group to another with an example.

Consider the groups (Z, f) and $(\{1, -1\},)$.If we define

$f : Z \to \{1, -1\}$ by $f(n) = \begin{cases} 1, \text{ if n is even} \\ -1, \text{ if n is odd,} \end{cases}$

then you can see that $f(a + b) = f(a).f(b) \;\; \forall a, b$ □ Z. What we have just seen is an example of a homomorphism, a function that preserves the algebraic structure of its domain.

Definition: Let $(G_1, *_1)$ and $(G_2, *_2)$ be two groups. A mapping f : G_1 — G_2 is said to be a group homomorphism (or just a homomorphism), if

$f(x *_1 y) = f(x) *_2 f(y) \;\; \forall x, \psi \in G_1$.

Note that a homomorphism f from G_1 to G_2 carries the product $x *_1 y$ in G_1 to the product

$f(x) *_2 f(y)$ in G_2.

Note: The word 'homomorphism' is derived from two Greek words 'homos', meaning 'link', and 'morphe', meaning 'form'.

Let us define two sets related to a given homomorphism.

Definition: Let $(G_1, *_1)$ and $(G_2, *_2)$ be two groups and $f : G_1 \in G_2$ be a homomorphism. Then we define

(i) the image of f to be the set

$\text{Im } f = \{ f(x) \; 1 \; x \in G_1 \}$.

(ii) the kernel of f to be the set

$\text{Ker } f = \{ x \in G_1 \mid f(x) = e_2 \}$, where e_2 is the identity of G_2.

Note that $\text{Im } f \subseteq G_2$, and $\text{Ker } f = f^{-1}(\{e_2\}) \in G_1$.

Example: Consider the two groups (R, +) and (R*,.). Show that the map exp : $(R, +) \to (R^*, .)$: $\exp(r) = e^r$ is a group homomorphism. Also find Im exp and Ker exp.

Solution: For any $r_1, r_2 \in R$, we know that

$\therefore \exp(r_1 + r_2) = \exp(r_1).\exp(r_2)$.`

Hence, exp is a homomorphism from the additive group of real numbers to the multiplicative group of non-zero real numbers.

Now, Im exp = $\{ \exp(r) \mid r \in R \} = \{ e^r \mid r \in R \}$,

Also, Ker exp = $\{ \rho \in R \mid e^r = l \} = \{ 0 \}$.

Note that examples takes the identity 0 of R to the identity 1 of R*. example also carries the additive inverse – r of r. to the multiplicative inverse of exp (r) .

Example: Consider the groups (R, +) and (C, +) and define f : (C, +) $\rightarrow$ (R, +) by f (x + iy) = x, the real part of x + iy. Show that f is a homomorphism. What are Im f and Ker f?

Solution: Take any two elements a + ib and c + id in C. Then,

$f((a + ib) + (c + id)) = f((a + c) + i(b + d)) = a + c = f(a + ib) + f(c + id)$

Therefore, f is a group homomorphism.

Imf = $\{ f(x + iy) \mid x, y \in R \} = \{ x \mid x \in R) = R$.

So, f is a surjective function

Ker f = $\{ x + iy \in C \mid f(x + iy) = 0 \} = \{ x + iy \in C \mid x = 0 \}$

= { iy | y E R }, the set of purely imaginary numbers.

Note that f carries the additive identity of C to the additive identity of R and (– z) to – f (z) , for any $z \in C$.

In Examples 1 and 2 we observed that the homomorphism's carried the identity to the identity and the inverse to the inverse. In fact, these observations can be proved for any group homomorphism.

Theorem: Let f ; $(G_1, *_1) \in (G_2, *_2)$ be a group homomorphism.

Then

(a) $f(e_l) = e_2$, where e_1 is the identity of G_1 and e_2 is the identity of G_2.

(b) $f(x^{-1}) = [f(x)]^{-1}$ for all x in G_1.

Proof: (a) Let $x \in G_1$. Then we have $e_1 * 1x = x$. Hence,

$f(x) = f(e_l *_1 x) = f(e_1) *_2 f(x)$, since f is a homomorphism.

But

$f(x) = e_2 *_2 f(x)$ in G_2.

Thus, $f(e_1) *_2 f(x) = e_2 *_2 f(x)$.

So, by the right cancellation law in G_2, $f(e_1) = e_2$.

(b) Now, for any $x \in G_1$, $f(x) *_2 f(x^{-1}) = f(x *_1 x^{-1}) = f(e_1)$ $= e_2$.

Similarly, $f(x^{-1}) *_2 f(x) = e_2$.

Hence, $f(x^{-1}) = [f(x)]^{-1}$ $\forall$ x E G_1.

Note that the converse of Theorem 1 is false. That is, if $f : G_1 \rightarrow G_2$ is a function such that $f(e_1) = e_2$ and $[f(x)]^{-1} = f(x^{-l}) = f(x^{-1})$ $\forall x \in G_1$, then f need not be a homomorphism.

For example, consider $f : Z \rightarrow Z : f(0) = 0$ and ,

$$f(n) = \begin{cases} n+1 \ \forall \ n > 0 \\ n-1 \ \forall \ n < 0 \end{cases}$$

Since $f(l+1) \in f(1) + f(l)$, f is not a homomorphism. But $f(e_1)$ $= e_2$ and $f(n) = -f(-n)$ $\forall n \in Z$.

Let us look at a few more examples of homomorphism's now. We can get one important class of homomorphism's from quotient groups.

Example: Let H $\trianglelefteq$ G. Consider the map $p : G \rightarrow G/H : p(x) =$ Hx. Show that p is **a** homomorphism. Also show that p is onto. What is Ker p?

Solution: For x, y $\in$ G, $p(xy) = Hxy = Hx\,Hy = p(x)\ p(y)$. Therefore, p is a homomorphism.

Now, Im p = $\{ p(x) \ (x \in G \} = (Hx \mid x \in G \} = G/H$. Therefore, p is onto.

Ker p = $\{ x \in G \mid p(x) = H)$. (Remember, H is the identity of G/H.)

$= (x \in G \mid Hx = H \}$

$= \{ x \in G \mid x \in\in H)$, by theorem.

= H.

In this example you can see that Ker p $\trianglelefteq$ G. You can also check that Theorem 1 is true here.

Example*:* Let H be a subgroup of a group G. Show that the map i : H → G, i (h) = h is a homomorphism. This function is called the inclusion map.

Solution: Since $i(h_1 h_2) = h_1 h_2 = i(h_1)\, i(h_2) \;\forall\; h_1, h_2 \in H$. i is a group homomorphism.

Let us briefly look at the inclusion map in the context of symmetric groups. Consider two natural numbers m and n, where m I n.

Then, we can consider $S_m \le S_n$, where any $\sigma \in$ Sm, written as

$\begin{pmatrix} 1 & 2 & .... & m \\ \sigma(1) & \sigma(2) & .... & \sigma(m) \end{pmatrix}$, is considered to be the same as

$\begin{pmatrix} 1 & 2 & .... & m & m+1 & .... & n \\ \sigma(1) & \sigma(2) & .... & \sigma(m) & m+1 & .... & n \end{pmatrix} \in S_n$, i.e., σ (k) = k for m + 1 □ k □ n.

Then we can define an inclusion map $i : S_m \to S_r$.

For example, under $i : S_3 \to S_4$, (1 2) goes to $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 4 \end{pmatrix}$.

We will now prove some results about homomorphisms .Henceforth, for convenience, we shall drop the notation for the binary operation, and write a * b as ab.

Now let us look at the composition of two homomorphisms. Is it a homomorphism? Let us see.'

Theorem: If $f : G_1 \to G_2$ and $g : G_2 \to G_3$ are two group homomorphisms, then the composite map $g \,.\, f : G_1 \to G_3$ is also a group homomorphism.

Proof: Let x. y $\in G_1$. Then

$g \circ f(xy) = g(f(x : y))$

$= g(f(x)\, f(y))$, since f is a homomorphism.

$= g (f (x)) \ g (f (g))$, since g is a homomorphism.

$= g \circ f (x) . g \circ f (y)$.

Thus g, f is a homomorphism.

Theorem: Let $f : G_1 \rightarrow G_2$ be a group homomorphism. Then

(a) Ker f is a normal subgroup of G_1.

(b) Im f is a subgroup of G_2.

Proof: (a) Since [(e_1) = e_2, $e_1 \in$ Ker f. $\therefore$ Ker f $\neq \phi$.

Now, if x, y $\in$ Ker f, then $f (x) = e_2$ and $f (y) = e_2$.

$\therefore f (xy^{-1}) = f (x) \ f (y^{-1}) = f (x) \ [f (y)]^{-1} = e_2$.

$xy^{-1} \in$ Ker f.

Therefore, by Theorem 1, Ker f $\leq G_1$. Now, for any $y \in G_1$ and x E Ker f,

$f (y^{-1}xy) = f (y^{-1}) \ f (x) f (y)$

$= [f (y)]^{-1}e_2 f (y)$, since $f (x) = e_2$ and by Theorem 1

$= e_2$.

$\therefore$ Ker f $\trianglelefteq G_1$.

(b) Im f $\neq \phi$, since f (e_l) E Im f.

Now, let $x_2, y_2 \in$ Im f. Then $\exists x_1, y_1 \in G_1$ such that $f (x_1) = x_2$ and $f (y_1) = y_2$.

$\therefore$ $x_2 y_2^{-1} = f (x_1) \ f (y_1^{-1}) = f (x_1 y_1^{-1}) \in$ Imf.

$\therefore$ Im f $\leq G_2$.

Using this result, we can immediately see that the set of purely imaginary numbers is a normal subgroup of C.

Consider $\phi : (R, +) \rightarrow (C^*,.) \ \phi (x) = \cos x + i \sin x$. We have seen that $\phi (x + y) = \phi (x) \phi (y)$, that is, ϕ is a group homomorphism. Now $\phi (x) = 1$ iff $x = 2\pi n$ for some $n \in Z$. Thus, by Theorem 3, Ker

$\phi = (2\pi n \mid n \in Z)$ is a normal subgroup of $(R. +)$. Note that this is cyclic, and 2n is a generator.

Similarly, Im ϕ is a subgroup of C*. This consists of all the complex numbers with absolute value 1, i.e., the complex numbers on the circle with radius 1 unit and centre $(0, 0)$.

You may have noticed that sometimes the kernel of a homomorphism is $\{e\}$ and sometimes it is a large subgroup. Does the size of the kernel indicate anything? We will prove that a homomorphism is 1 – 1 iff its kernel is $\{e\}$.

Theorem: Let $f : G_1 \to G_2$ be a group homomorphism. Then f is injective iff Ker f = $\{e_1\}$, where e_1 is the identity element of the group G_1.

Proof: Firstly, assume that f is injective. Let $x \in$ Ker f. Then $f(x) = e_2$, i.e., $f(x) = f(e_1)$. But f is 1 – 1. $\therefore x = e_1$.

Thus, Kerf = $\{e_1\}$.

Conversely, suppose Ker f = $\{e_1]$. Let $x, y \in G_1$ such that

$f(x) = f(y)$. Then $f(xy^{-1}) = f(x)\ f(y^{-1})$

$$= f(x)\ [f(y)]^{-1} = e_2.$$

$\therefore \quad xy^{-1} \in$ Ker f = $\{e1\}$. $\quad \therefore xy^{-1} = e_1$ and $x = y$.

This shows that f is injective.

So, by using Theorem 4, we can immediately say that any inclusion $i : B \to G$ is 1-1, since
Ker i = $\{e\}$.

Let us consider another example.

Example: Consider the group T of translations of R^2. We define a map $\phi : (R^2 +) \to (T, o)$ …. $(a, b) = f_{a,\ b}$. Show that ϕ is an onto homomorphism, which is also 1-1.

Solution: For (a, b), (c, d) in R*, we have seen that

$f_{a+c,h+d} = f_{a,b}\ o\ f_{c.d}$

Notes

$\therefore \quad \phi((a, b) + (c, d)) = \phi(a, b) \circ \phi(c, d)$.

Thus, ϕ, is a homomorphism of groups.

Now, any element of T is -f (a, b) . Therefore, ϕ is surjective. We now show that ϕ is also injective.

Let (R, b) $\in$ Ker ϕ. Then \$ (a, b) = $f_{0,0}$

i.e., $f_{a,b} = f_{0,0}$

$\therefore \; f_{a,b}(0, 0) = f_{0,0}(0, 0)$,

i.e., (a, b) = (0, 0)

$\therefore$ Ker ϕ = { 1 (0, 0) }

$\therefore$ ϕ is 1-1.

So we have proved that f is a homomorphism, which is bijective.

And now let us look at a very useful property of a homomorphism that is surjective.

Theorem: Iff : $G_1 \rightarrow G_2$ is an onto group homomorphism and S is a subset that generates G_1, then f (S) generates G_2.

Proof: We know that

G_1 = < S > = { | m $\in$ N, $x_1 \in$ S, $r_1 \in$ Z for all i) . We will show that

G_2 = < f (S) >

Let x $\in G_2$, Since f is surjective, there exists y $\in G_1$ such that f (y) = x. Since y $\in G_1$, y = for some m $\in$ N, where xi $\in$ S and ri $\in$ Z, $\leq 1 \leq i \leq$ m.

Thus,x = f (y) = = since f is a homomorphism.

$\Rightarrow$ x $\in$ < f (S) >. since f (x_1) $\in$ f (S) for every i = 1, 2, ..., r.

Thus G_2 = < f (S) >.

So far you have seen examples of various kinds of homomorphisms-injective, surjective and bijective. Let us now look at bijective homomorphism in particular.

Check Your progress-1

1. Let H be a subgroup of a Group G. Then $H \to G$, $i(h) = h$ is a homomorphism. This function is called the ....

   (a) inclusion map   (b) normal function

   (c) cyclic   (d) abelian

2. gof (x, y) is equal to:

   (a) gof (x) . gof (y)   (b) gof (x) + gof (y)

   (c) gof (x^{-1}) .gof (y^{-1})   (d) gof (x) . gof (y^{-1})

1.3 ISOMORPHISMS

Definition: Let G_1 and G_2 be two groups. A homomorphism $f : G_1 \to G_2$ is called an isomorphism if f is 1-1 and onto.

In this case we say that the group G_1 is isomorphic to the group G_2 or G_1 and G_2 are isomorphic. We denote this fact by $G_1 \approx G_2$.

An isomorphism of a group G onto itself is called an automorphism of G. For example, the identity' function IG : $G \to G$: $I_G(x) = x$ is an automorphism.

Note: The word 'isomorphisms' is derived from the Greek word
'ISOS' meaning
'equal'.

Let us look at another example of an isomorphism.

Exampl*e:* Consider the set G = $\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| a, b \in R \right\}$.

Then G is a group with respect to matrix addition.

Show that $f : G \to C : f\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) = a + ib$ is an isomorphism.

Solution: Let us first verify that f is a homomorphism. Now, for any two elements

$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $\begin{bmatrix} c & d \\ -d & c \end{bmatrix}$ in G,

Notes

$$r\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right) = f\begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix} = (a+c) + i(b+d)$$

= (a + ib) + (c + id)

$$= f\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) + f\left(\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right)$$

Therefore, f is a homomorphism.

Now, Ker f = $f = \left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| a+ib=0\right\} = \left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| a+0, b=0\right\} = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$

Therefore, by Theorem 4, f is 1-1.

Finally, since Im f = C. f is surjective

Therefore, f is an isomorphism.

We would like to make an important remark now.

Remark: If G_1 and G_2 are isomorphic groups, they must have the same algebraic structure and satisfy the same algebraic properties. For example, any group isomorphic to a finite group must be finite and of the same order. Thus, two isomorphic groups are algebraically indistinguishable systems.

The following result is one of the consequences of isomorphic groups being algebraically alike

Theorem: If $f : G \rightarrow H$ is a group isomorphism and $Y \in G$, then $< x >$ $; < f \ (x) >$,

Therefore.

(i) if s is of finite order, then o (x) = o (f (s)) .

(ii) if x is of infinite order, so is f (x) .

Proof: If we restrict f to any subgroup K of G, we have the function f | K ; $K \rightarrow f (K)$, Since f is bijective, sc is its restriction f | k ; k ; f (K) for any subgroup K of G. In particular, for any $x \in G$,

$< x > ; f (< x >) = < f (x) >$,

Now if x has finite order, then o (x) = o (< x >) = o (< f (x) >) = o (f (x)), proving (i)

To prove (ii) assume hat x is of infinite order. Then < x > is an infinite group.

Therefore, < f (x) > is an infinite group, and hence, f (x) is of infinite order. So, we have proved (ii) .

Example: Show that (R*,.) is not isomorphic to (C*,.) .

Solution: Suppose they are isomorphic, and f : C* — R* is an isomorphism. Then

o (i) = o (f (i)) , by Theorem 6, Now o (i) = 4. ∴ o (f (i)) = 4.

However, the order of any real number different from ±1 is infinite: and o (1) = 1, o (–1) = 2.

So we reach a contradiction. Therefore, our supposition must be wrong. That is, R* and C* are not isomorphic.

You must have noticed that the definition of an isomorphism just says that the map is bijective, i.e., the inverse map exists. It does not tell us any properties of the inverse. The next result does so.

Theorem: If $f : G_1 \to G_2$ is an isomorphism of groups, then $f^{-1} : G_2 \to G_1$ is also an isomorphism.

Proof: You know that f^{-1} is bijective. So, we only need to show that f^{-1} is a homomorphism. Let a', b' $\in G_2$ and a = f^{-1} (a') , b = f^{-1} (b') . Then f (a) = a' and f (b) = b'.

Therefore, f (ab) = f (a) f (b) = a'b'. On applying f^{-1}, we get

f^{-1} (a'b') = ab = f^{-1} (a') f^{-1} (b') , Thus,

f^{-1} (a'b') = f^{-1} (a') f^{-1} (b') for all a', b' $\in G_2$.

Hence, f^{-1} is an isomorphism.

From Theorem 7 we can immediately say that

$\phi^{-1} : T \to R^2 : \phi^{-1} (f_{a,b,}) = (a, b)$ is an isomorphism.

Notes

Theorem says that if G_I ; G_2, then G_2 ; G_1. We will be using this result quite often.

Check Your progress-2

3. An isomorphism of a group G onto itself is called an ___________of G.

   a. automorphism

   b. isomorphic

   c. homomorphism

   d. Non of the above

4. If $f : G_1 \rightarrow G_2$ is an isomorphism of groups, then $f^{-1} : G_2 \rightarrow G_1$ is also an __________.

   a. automorphism

   b. isomorphic

   c. homomorphism

   d. Non of the above

1.4 GROUP ISOMORPHISM

In abstract algebra, a group isomorphism is a function between two groups that sets up a one-to-one correspondence between the elements of the groups in a way that respects the given group operations. If there exists an isomorphism between two groups, then the groups are called isomorphic. From the standpoint of group theory, isomorphic groups have the same properties and need not be distinguished.

Definition and Notation

Given two groups (G, *) and (H,) , a group isomorphism from (G, *) to (H,) is a bijective group homomorphism from G to H. Spelled out, this means that a group isomorphism is a bijective function $f : G \rightarrow H$ such that for all u and v in G it holds that

$f (u * v) = f (u) \; f (v)$.

The two groups (G, *) and (H,) are isomorphic if an isomorphism exists. This is written:

(G, *) ≅ (H,)

Often shorter and more simple notations can be used. Often there is no ambiguity about the group operation, and it can be omitted:

G ≅ H

Sometimes one can even simply write G = H. Whether such a notation is possible without confusion or ambiguity depends on context. For example, the equals sign is not very suitable when the groups are both subgroups of the same group.

Conversely, given a group (G, *) , a set H, and a bijection f : G → H, we can make H a group
(H,) by defining

f (u) f (v) = f (u * v) .

If H = G and = * then the bijection is an automorphism (q.v.)

Intuitively, group theorists view two isomorphic groups as follows: For every element g of a group G, there exists an element h of H such that h 'behaves in the same way' as g (operates with other elements of the group in the same way as g) . For instance, if g generates G, then so does h. This implies in particular that G and H are in bijective correspondence. So the definition of an isomorphism is quite natural.

An isomorphism of groups may equivalently be defined as an invertible morphism in the category of groups, where invertible here means has a two-sided inverse.

Examples:

1. The group of all real numbers with addition, (ϒ, +) , is isomorphic to the group of all positive real numbers with multiplication ($ϒ^+$, ×) :

(ϒ, +) (ϒ+, ×)

via the isomorphism

$f (x) = e^x$

Notes

(see exponential function) .

2. The group ′ of integers (with addition) is a subgroup of □, and the factor group ϒ/′ is isomorphic to the group S^1 of complex numbers of absolute value 1 (with multiplication) :

ϒ/′ ≅ S^1

An isomorphism is given by

f (x + ′) = $e^{2\pi x1}$

for every x in ϒ.

3. The Klein four-group is isomorphic to the direct product of two copies of $′_2$ = ′/2′

(see modular arithmetic) , and can therefore be written $′_2 \cdot ′_2$. Another notation is Dih_2, because it is a dihedral group.

4. Generalizing this, for all odd n, Dih_{2n} is isomorphic with the direct product of Dih_n and Z_2.

5. If (G, *) is an infinite cyclic group, then (G, *) is isomorphic to the integers (with the addition operation) . From an algebraic point of view, this means that the set of all integers (with the addition operation) is the 'only' infinite cyclic group.

Some groups can be proven to be isomorphic, relying on the axiom of choice, but the proof does not indicate how to construct a concrete isomorphism.

1. The group (ϒ, +) is isomorphic to the group (≤, +) of all complex numbers with addition.

2. The group ($≤^*$, ·) of non-zero complex numbers with multiplication as operation is isomorphic to the group S^1 mentioned above.

Properties

The kernel of an isomorphism from (G, *) to (H,) , is always { e_G } where e_G is the identity of the group (G, *) If (G, *) is isomorphic to (H,) , and if G is abelian then so is H.

If (G, *) is a group that is isomorphic to (H,) [where f is the isomorphism], then if a belongs to G and has order n, then so does f (a) .

If (G, *) is a locally finite group that is isomorphic to (H,) , then (H,) is also locally finite.

We also go through that 'group properties' are always preserved by isomorphisms.

Cyclic Groups

All cyclic groups of a given order are isomorphic to $('_n, +_n)$.

Let G be a cyclic group and n be the order of G. G is then the group generated by < x > = { e,x,...,
x^{n-1} } . We will show that

$$G \cong ('_n, +_n)$$

Define

$\phi : G \to '_n = \{ 0, 1,..., n - 1 \}$, so that $\phi (x^a) = a$. Clearly, ϕ is bijective.

Then

$\phi (x^a . x^b) = \phi (x^{a+b}) = a + b = \phi (x^a) +_n \phi (x^b)$ which proves that $G \cong '_n$, +n.

Consequences

From the definition, it follows that any isomorphism $f : G \to H$ will map the identity element of G to the identity element of H,

$$f (e_G) = e_H$$

that it will map inverses to inverses,

$$f (u^{-1}) = [f (u)]^{-1}$$

and more generally, nth powers to nth powers,

$$f (u^n) = [f (u)]^n$$

for all u in G, and that the inverse map $f^{-1} : H \to G$ is also a group isomorphism.

The relation "being isomorphic" satisfies all the axioms of an equivalence relation. If f is an isomorphism between two groups G and H, then everything that is true about G that is only related to the group structure can be translated via f into a true ditto statement about H, and vice versa.

Check Your progress-3

5. Let $f : G_1 \to G_2$ be a group homomorphism thus her f is a .... of G.

   (a) subgroup (b) normal

   (c) cyclic (d) abelian

6. If (G, *) is isomorphic to (H,) , and if G is abelian then so is.

   (a) H (b) G

   (c) Both H & G (d) Non of the above

1.5 LET US SUM UP

In this unit we have discussed the definition and example of a group homomorphism. Let $f : G_1 \to G_2$ be a group homomorphism. Then $f (e_1) = e_2$, $[f (x)]^{-1} = f (x^{-1})$, Im $f \leq G_2$, Ker f G_1.

We have also discussed a homomorphism is 1-1 iff its kernel is the trivial subgroup. The definition and examples of a group isomorphism. Two groups are isomorphic if they have exactly the same algebraic structure.

Lastly we have discussed the composition of group homomorphisms (isomorphisms) is a group homomorphism (isomorphism) .

1.6 KEYWORDS

1. **Homomorphism:** Homomorphism is derived from two Greek words 'homos', meaning 'link', and 'morphe', meaning 'form'.

2. Inclusion map: Let H be a subgroup of a group G. Show that the map i: H → G, i (h) = h is a homomorphism. This function is called the **inclusion map.**

1.7 QUESTIONS FOR REVIEW

1. Show that f : (R*,.) → (R, 4) : f (x) = inx, the natural logarithm of x, is a group homomorphism. Find Ker f and Im f also.
2. Is f : (GL_3 (R) 3,) → (w*,.) : f (A) = det (A) a homomorphism? If so, obtain Ker f and Im f.
3. Define the natural homomorphism p from S_3 to S_3/A_3. Does (1 2) E Ker p? Does (1 2) E Im p?
4. Let S = [z ∈ C| |z| = 1 } .

   Define f : (R, +) → (S,.) L f (x) = e^{Inx}, where n is a fixed positive integer. Is f a homomorphism? If so find Ker f.
5. Define Inclusion map by giving a suitable example.

1.8 SUGGESTED READINGS AND REFERENCES

1. Thomas W Judson (2013) . *Abstract Algebra: Theory and Applications*. Orthogonal Publishing.
2. Paul B. Garrett (2007) . *Abstract Algebra*. Chapman and Hall/CRC.
3. Vijay K Khanna (2017) .*A Course in Abstract Algebra Fifth Edition*. Vikas Publishing House
4. LALJI PRASAD (2016) . *Modern Abstract Algebra*. Paramount Publication
5. Stephen Lovett (2016) . *Abstract Algebra: Structures and Applications*. Chapman and Hall/CRC

1.9 ANSWERS TO CHECK YOUR PROGRESS

Notes

1. (a) (answer for Check your Progress-1 Q.1)
2. (a) (answer for Check your Progress-1 Q.2)
3. (a) (answer for Check your Progress-2 Q.3)
4. (b) (answer for Check your Progress-2 Q.4)
5. (b) (answer for Check your Progress-3 Q.5)
6. (a) (answer for Check your Progress-3 Q.6)